point on (t_0, t_1). Following Schmitendorf and Citron[3] or Breakwell and Ho,[4] there is a conjugate point at t^* for this example if there exists a nontrivial solution of

$$\dot{x} = -\lambda \quad \text{(3a)}$$

$$\dot{\lambda} = \tfrac{1}{4}x \quad \text{(3b)}$$

$$x(t_1) = 0 \quad \text{(3c)}$$

with $x(t^*) = 0$. The solution of (3) is

$$x(t) = A \sin\tfrac{1}{2}(t - t_1)$$

$$\lambda(t) = -\tfrac{1}{2}A \cos\tfrac{1}{2}(t - t_1)$$

Therefore, there is a conjugate point at $t^* = t_1 - 2\pi$. Hence, if $0 < t_1 < 2\pi$, there is not a conjugate point on $[0, t_1]$ and the second variation is positive definite. If $t_1 > 2\pi$, there is a conjugate point and the solution (1) is not optimal.

Lastman and Tapley compare their result with the Riccati differential equation method presented by Athans and Falb.[5] However, the method presented by Athans and Falb applies only to problems with the final time t_1 specified and the final state $X(t_1)$ unspecified. The previous example does not fulfill the second condition, and the Riccati method presented in Athans and Falb cannot be used.

If the Riccati transformation method is to be used for the example problem, the technique presented by McReynolds and Bryson[6] and extended by Schmitendorf and Citron[7] must be followed. For the example problem, there is a conjugate point at t^* if $P - RQ^{-1}R'$ or RQ^{-1} becomes infinite at t^* where

$$\dot{P} = P^2 + \tfrac{1}{4} \qquad P(t_1) = 0 \quad \text{(4a)}$$

$$\dot{R} = PR \qquad R(t_1) = 1 \quad \text{(4b)}$$

$$\dot{Q} = R^2 \qquad Q(t_1) = 0 \quad \text{(4c)}$$

On solving (4), one finds that

$$P - RQ^{-1}R' = \tfrac{1}{2} \cot\tfrac{1}{2}(t_1 - t)$$

$$RQ^{-1} = -\tfrac{1}{2} \csc\tfrac{1}{2}(t_1 - t)$$

Since $P - RQ^{-1}R'$ and RQ^{-1} become infinite at $t^* = t_1 - 2\pi$, the second variation is positive definite and the solution (1) optimal if $0 < t_1 < 2\pi$.

If the conditions at t_1 in the example problem are changed to t_1 fixed and $X(t_1)$ unspecified, the extremal solution is

$$X(t) = 0 \qquad P(t) = 0 \qquad \cos(t_1/2) \neq 0$$

For this problem there is a conjugate point at t^* if there exists a nontrivial solution of

$$\dot{x} = -\lambda \quad \text{(5a)}$$

$$\dot{\lambda} = \tfrac{1}{4}x \quad \text{(5b)}$$

$$\lambda(t_1) = 0 \quad \text{(5c)}$$

with $x(t^*) = 0$. The solution of (5) is

$$x(t) = A \cos\tfrac{1}{2}(t - t_1) \qquad \lambda(t) = \tfrac{1}{2}A \sin\tfrac{1}{2}(t - t_1)$$

and there is a conjugate point at $t^* = t_1 - \pi$. The second variation is positive definite and the solution optimal if $0 < t_1 < \pi$. Thus the conclusions of Lastman and Tapley apply to the example problem with the end conditions modified to t_1 given and $X(t_1)$ unspecified but are not true if t_1 is given and $X(t_1) = 0.2$.

References

[1] Lastman, G. J. and Tapley, B. D., "A Test for the Sign of the Second Variation," *AIAA Journal*, Vol. 5, No. 9, Sept. 1967, pp. 1682–1683.

[2] Bliss, G. A., *Lectures on the Calculus of Variations*, University of Chicago Press, Chicago, Ill., 1946, pp. 253–257.

[3] Schmitendorf, W. E. and Citron, S. J., "On the Conjugate Point Condition for a Class of Differential Games," A.A.&E.S. Rept. 67-11, 1967, Purdue Univ., Lafayette, Ind.

[4] Breakwell, J. V. and Ho, Y. C. "On the Conjugate Point Condition for the Control Problem," *International Journal of Engineering Science*, Vol. 2, No. 6, March 1965, pp. 565–579.

[5] Athans, M. and Falb, P. L., *Optimal Control*, McGraw-Hill, New York, 1966, pp. 756–766.

[6] McReynolds, S. R. and Bryson, A. R., "A Successive Sweep Method for Solving Optimal Programming Problems," *Joint Automatic Control Conference*, Rennselaer Polytechnic Institute, Troy, N.Y., 1965.

[7] Schmitendorf, W. E. and Citron, S. J. "On the Applicability of the Riccati Transformation Technique for Solution of Optimal Control Problems," A.A.&.E.S. Rept. 67-10, 1967, Purdue Univ., Lafayette, Ind.

Reply by Authors to W. E. Schmitendorf

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Schmitendorf has pointed out an error in the example problem of our paper. This error is due to an incorrect problem formulation in the given example. In our paper, a method was developed for testing the sign of the second variation. The test required that the matrix

$$S(\tau,t_1) = M_1{}^T(t_1)A(t_1)M_1(t_1) - M_2{}^T(t_1)M_1(t_1)$$

be positive definite. The matrix $A(t_1)$ contains the effects of second variations in the terminal conditions. For the example problem in our paper, $A(t_1) \equiv 0$ so that changes in the terminal conditions could not influence $S(\tau,t_1)$. In such cases we can reformulate the problem so that the terminal constraints $L[X(t_1),t_1] = 0$ are converted into a penalty factor $\frac{1}{2}KL^TL$ (K is a non-negative constant) which is then added to the original performance index. We illustrate this by the following revised problem: minimize

$$\Gamma = \frac{1}{2} K[X(t_1) - 0.2]^2 + \frac{1}{2}\int_{t_0}^{t_1} \left(U^2 - \frac{1}{4}X^2\right) dt$$

subject to $\dot{X} = U$, $t_0 = 0$, $X(t_0) = 0$, t_1 specified, and $K \geq 0$. The extremal solution is $X(t) = c\sin(t/2)$, $P(t) = (-c/2)\cos(t/2) = -U(t)$, $c = 0.2\cos\delta/\sin(t_1/2 + \delta)$, $\cos\delta = 2K(4K^2 + 1)^{-1/2}$, $\sin\delta = (4K^2 + 1)^{-1/2}$, $\sin(t_1/2 + \delta) \neq 0$. Thus $0 < \delta \leq \pi/2$ since $\infty > K \geq 0$. On applying the condition for the second variation to be non-negative we obtain $S(\tau,t_1) = 4K\sin^2 y + 2\sin y(\cos y)$, where $y = (t_1 - \tau)/2$. Rewriting $S(\tau,t_1)$ in terms of δ, we obtain

$$S(\tau,t_1) = 2\sin y \sin(y + \delta)/\sin\delta$$

For $0 < \delta \leq \pi/2$, $\sin\delta > 0$; hence, $S(\tau,t_1) > 0$ if $0 < y + \delta < \pi$. Setting $\tau = 0$ we have that $0 < t_1 < 2\pi - 2\delta$ as the condition for a positive second variation. If $K = 0$ $(\delta = \pi/2)$ $X(t_1)$ is completely unspecified, and hence $0 < t_1 < \pi$, in agreement with the result given by Schmitendorf. As $K \to \infty$, $\delta \to 0$, and the range for positive second variation becomes

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$0 < t_1 < 2\pi$. $K \to \infty$ corresponds to the case where $X(t_1)$ is specified.

The Riccati equation for this revised problem is $\dot{W} = W^2 + \frac{1}{4}$, $W(t_1,t_1) = K$, so that $W(\tau,t_1) = -\frac{1}{2}\tan[-\tan^{-1}(2K) + y]$. Thus $W(0,t_1)$ is finite if $0 < t_1 < \pi + 2\tan^{-1}(2K)$. For $K = 0$ this gives $0 < t_1 < \pi$; for $K \to \infty$, $0 < t_1 < 2\pi$.

Comment on "A Combined Visual and Hot-Wire Anemometer Investigation of Boundary-Layer Transition"

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KNAPP and Roache[1] discuss the streakline pattern of the disturbed boundary layer on ogive nose cylinders. With regard to the rolling-up process of the streak-lines in their transition region R_1, they referred to the paper of Hama[2] who calculated the streakline pattern of a free shear layer perturbed by the neutral disturbance due to the linearized stability theory. Hama found that the streaklines roll up, as if to indicate that the flow develops into vortices. Hama assumed that in the perturbed flow there was no vorticity concentration, and it followed that the rolling-up of streaklines cannot constitute a positive identification of the presence of discrete vortices, and Knapp and Roache[1] argued similarly as well.

This statement of Hama, however, was incorrect, since in the perturbed flow used by him, local concentrations of vorticity existed which essentially corresponded to a one-row vortex street configuration as shown in Refs. 3–5. For amplified disturbances, which were investigated by Knapp and Roache, the rolling-up process of streaklines has been calculated in Refs. 5 and 6. The agreement of the theoretical results with the observed smoke pattern in experiments[7] was good. Furthermore, it was found in Refs. 5 and 6 that the rolling-up process of the streaklines corresponds to a local concentration of vorticity.

I therefore suppose that the streakline pattern observed by Knapp and Roache in their region R_1 will, in fact, indicate a concentration of vorticity, since the streaklines look similar to those observed in free shear layers.[7] Another question, however, is whether these concentrations of vorticity can be characterized as "discrete vortices." I think that this is a question of definition. Surely, these local concentrations of vorticity are not of the type found in a potential vortex or in a Hamel-Oseen vortex. But I should prefer to denote an essential local concentration of vorticity as a discrete vortex, since the effect of a local concentration of vorticity is similar to that of a discrete vortex due to the induction law.

References

[1] Knapp, C. F. and Roache, P. J., "A Combined Visual and Hot-Wire Anemometer Investigation of Boundary-Layer Transition," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 29–36.

[2] Hama, F. R., "Streaklines in a Perturbed Shear Flow," *The Physics of Fluids*, Vol. 5, No. 6, June 1962, pp. 644–650.

[3] Michalke, A., "On the Inviscid Instability of the Hyperbolic-Tangent Velocity Profile," *Journal of Fluid Mechanics*, Vol. 19, 1964, pp. 543–556.

[4] Michalke, A., "Vortex Formation in a Free Boundary Layer According to Stability Theory," *Journal of Fluid Mechanics*, Vol. 22, 1965, pp. 371–383.

[5] Michalke, A., "On Spatially Growing Disturbances in an Inviscid Shear Layer," *Journal of Fluid Mechanics*, Vol. 23, 1965, pp. 521–544.

[6] Michalke, A. and Freymuth, P., "Separated Flows," *AGARD Conference Proceedings*, No. 4, Part 2, 1966, pp. 575–595.

[7] Freymuth, P., "On Transition in a Separated Laminar Boundary Layer," *Journal of Fluid Mechanics*, Vol. 23, 1965, pp. 683–704.

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Reply by Author to A. Michalke

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WE genuinely appreciate Michalke's Comment[1] and this opportunity to discuss a point that has clouded the literature on boundary-layer transition for years. A long-standing semantic confusion exists between Michalke[1,2] and Hama[3,4] over the imprecise use of the words "vorticity maximum" and "vorticity concentration." Both use the word "concentration" with what they want to characterize as a discrete vortex. But Michalke[2,5] has used the words "maximum," "extremum," "peak," and "concentration" interchangeably, whereas Hama[3,4] clearly has used "maximum" and "concentration" with different meanings. Hama[3] certainly did not "pretend"[1] that there was no vorticity concentration; he plotted out both the mean and the fluctuating vorticity distributions. Also, he says "... the flow ... does not explicitly exhibit singularities. The mean vorticity ... as well as the fluctuating vorticity ... are both broadly distributed in y direction ... and the latter fluctuates sinusoidally; there is [sic] no higher harmonics in the fluctuations or no vorticity concentrations in this flow." Hama uses "concentration" and "discrete vortex" only in the sense of a singularity, at least in Ref. 3. But Michalke[5] quite successfully interprets free shear layer instability in terms of vortex induction, and so refers to the smooth vorticity extrema as "concentrations."

This leads to the problem of definition of a "discrete vortex," the problem being that no singularity can exist in viscous flow; and by Hama's criteria in Ref. 3, no discrete vortex can exist. This is clearly an inadequate definition, as recognized by all parties, including Hama.[4] Thus exists the present argument (also treated by Hama and Nutant,[4] Klebanoff, Tidstrom, and Sargent,[6] and Kovasznay, Komoda, and Vasudeva[7]) over whether the discrete vortex is formed before or after the three-dimensional deformation of the R_1 wave front.

Consider other definitions. One textbook[8] simply states that "a region containing vorticity is called a vortex." (Thus, the stable laminar boundary layer itself is a vortex.) But it seems that everyone involved envisions "vortex" to require some closed streamlines and some sort of vorticity maximum. Now the so-called Tollmien-Schlichting waves exhibit these characteristics (see Fig. 16.14 of Schlichting[9]), but in customary usage they have not been called vortices. Somehow, the vorticity has not been concentrated enough to warrant the characterization as a discrete vortex. And although vague agreement has been reached among different investigators on the distinction, the fact is that no one has ever quantified his criterion.

There has been another criterion widely used by experimentalists[6,7] to distinguish wavy disturbances of the Toll-

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